ON THE NUMBER OF REAL ROOTS OF RANDOM BERNSTEIN POLYNOMIALS

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Abstract. In this paper we study some properties of the number of real roots of random Bernstein polynomials in a given interval such as: expectation, central limit theorems, universality of expectation and persistence probability (no zero-crossings). In the proofs, we exploit the analogous behaviour between the number of roots of Bernstein polynomials and the one of elliptic model.

1. Introduction

Study the number of real roots of random polynomials is a topic of much interest and long history in probability theory. Some initial works are dated back to 1782 by Waring (see [9]). We refer the readers to two standard monographs by Bharucha-Reid et al [4] and Farahmand [10] for the motivation.

Various models have been proposed to investigate. Consider a sequence \( \{a_0, a_1, \ldots\} \) (and also \( b_i \)'s) of i.i.d random variables, let us recall some well-known models:

- Kac’s model: \( f_n(x) = a_0 + a_1 x + \ldots + a_n x^n \).
- Elliptic model or Kostlan-Shub-Smale model:

\[
\tag{1}
f_n(x) = \sqrt{\binom{n}{0}} a_0 + \sqrt{\binom{n}{1}} a_1 x + \ldots + \sqrt{\binom{n}{n}} a_n x^n.
\]

- Weyl model: \( f_n = \sqrt{\frac{1}{0!}} a_0 + \sqrt{\frac{1}{1!}} a_1 x + \ldots + \sqrt{\frac{1}{n!}} a_n x^n \).
- Trigonometric model: \( f_n(x) = \sum_{i=0}^{n} a_i \cos(ix) + b_i \sin(ix) \).
- Classical trigonometric model: \( f_n(x) = \sum_{i=0}^{n} a_i \cos(ix) \).

With these models, the authors focus on the following properties of the number of real roots:

- **Expectation.** For random Kac polynomials, some upper and lower bounds for the expectation of number of real roots of are given by Littlewood and Offord [15] in the case where the coefficients \( a_i \)'s are i.i.d with standard normal distribution. And then Kac provided the celebrated Kac-Rice formula to calculate explicitly the expectation of number of real roots on any interval, that is

\[
\mathbb{E}(N_n[a, b]) = \int_a^b dt \int_{\mathbb{R}} |y| p_{f_n(t), f'_n(t)}(u, y) dy,
\]

where \( N_n[a, b] \) is the number of real roots of the polynomial \( f_n \) on the interval \([a, b]\) and \( p_{f_n(t), f'_n(t)} \) stands for the joint density function of the random vector \((f_n(t), f'_n(t))\). The
Kac-Rice formula only works in the case the coefficients $a_i$'s have continuous distribution. Especially, when $(a_i)_{i.i.d} \sim \mathcal{N}(0, 1)$, one has

\[ \mathbb{E}_{\text{Kac}}(N_n[\mathbb{R}]) = \frac{2}{\pi} \log n + C + o(1), \quad \mathbb{E}_{\text{Elliptic}}(N_n[\mathbb{R}]) = \sqrt{n}, \quad \mathbb{E}_{\text{Weyl}}(N_n[\mathbb{R}]) = \sqrt{n} \left( \frac{2}{\pi} + o(1) \right), \]

\[ \mathbb{E}_{\text{trigo}}(N_n[0, 2\pi]) = \sqrt{\frac{(2n+1)(2n+2)}{3}}, \quad \mathbb{E}_{\text{classic}}(N_n[0, 2\pi]) = \frac{2n+1}{\sqrt{3}} + C + o(1). \]

It is worth to notice that for elliptic model, Edelman and Kostlan [9] gave a beautiful geometric meaning of the expectation.

• Central limit theorems. The Kac-Rice formula is then generalized to calculate higher moments of the number of roots. Once the expectation and variance are given, a natural question is whether one has central limit theorems for the number of roots. A breakthrough result is due to Maslova [18] in 1975, where the CLT for Kac’s model is proven not only for continuous case but also for discrete case. And then there is a gap in this direction until very recently. In 2011, the next CLT result for trigonometric case is given by Granville and Wigman [12] by using the approximation of stationary Gaussian processes. Using Stein-Malliavin method, Azaïs et al [2, 3] prove the CLT for both trigonometrics cases and then Dalmao [5] provides the proof for elliptic model. Recently, the CLT for the number of roots of Weyl polynomials is given by Do and Vu [8] by comparing the correlation functions corresponding to the zero-points processes.

• Universality. The first universality result is given by Ibragimov and Maslova [13] for the number of real roots of Kac polynomials. They state that for any distribution of $a_i$’s such that $\mathbb{E}a_i = 0, \mathbb{E}a_i^2 = 1$ and $\mathbb{E}|a_i|^{2+\epsilon} < \infty$ for some $\epsilon > 0$, one has

\[ (2) \quad \mathbb{E}_{\text{Kac}}(N_n[\mathbb{R}]) = \left( \frac{2}{\pi} + o(1) \right) \log n, \quad \text{Var}_{\text{Kac}}(N_n[\mathbb{R}]) = \left[ \frac{4}{\pi} \left( 1 - \frac{2}{\pi} \right) + o(1) \right] \log n. \]

Here they use the approximation of the number of real roots by the number of sign changes. This method is recently used also by Flasche [11] to prove the universality of the expectation of the number of roots of trigonometric model.

Tao and Vu [21] propose another method based on the comparing correlation functions to prove the universality phenomenon for algebraic polynomials (Kac, elliptic, Weyl). Extending this method, in [19], the authors give an interesting result for Kac model,

\[ (3) \quad \mathbb{E}_{\text{Kac}}(N_n[\mathbb{R}]) = \frac{2}{\pi} \log n + C_{\text{dist}} + o(1), \]

where the constant $C_{\text{dist}}$ depends only on the distribution of the coefficients $a_i$’s. See also [14] for other works in this direction.

• Persistence probability (no zero-crossings). The persistence probability is the probability that on a given interval, the polynomial has no zero, or equivalently the sign of it does not change. For Kac polynomials, Littlewood and Offord [16] provide the upper bound of order $(\log n)^{-1}$. In [7], Dembo et al give the precise asymptotic formula $n^{-b+o(1)}$. Later on, Schehr and Majumdar [20] propose conjectures for persistence probability of elliptic and Weyl polynomials. Then Dembo and Mukherjee [6] confirm the positive answer for elliptic model. The method used in [6] requires the positivity of the autocorrelation function of random polynomials. This property holds for algebraic models, but trigonometric models. So the persistence probability for these cases is still unknown.
In this paper, we are interested in the random Bernstein polynomials
\begin{equation}
F_n(x) = \sum_{i=0}^{n} \sqrt{\binom{n}{i}} a_i x^i (1 - x)^{n-i},
\end{equation}
where the random variables $a_i$'s are i.i.d. This model is proposed by Armentano and Dedieu [1] with motivation from the study of Computer-aided Geometric Design on Bézier curves. They study the expected number of real roots in a given interval $[\alpha, \beta] \subset \mathbb{R}$ and state that in the Gaussian case,
\begin{equation}
\mathbb{E}(N_{F_n}\alpha, \beta]) = \frac{\sqrt{n}}{\pi}(\arctan(2\beta - 1) - \arctan(2\alpha - 1)).
\end{equation}

Here we investigate the other properties as: CLT, universality and persistence probability. Our main result is the following theorem.

**Theorem 1.1.** Consider a sequence of i.i.d random variables $\{a_0, a_1, \ldots\}$ of zero mean, unit variance, finite $(2 + \epsilon)$-moment for some $\epsilon > 0$, satisfying $\mathbb{P}(a_i = 0) = 0$. Define the Bernstein polynomial as in (4) and denote $N_{F_n}\alpha, \beta]$ (for short $N_n$) the number of real roots in a given interval $[\alpha, \beta]$. The following statements hold.

i. If $a_i \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$, then
\begin{equation}
\frac{N_{F_n}\alpha, \beta] - \mathbb{E}(N_{F_n}\alpha, \beta])}{n^{1/4}} \overset{d}{\to} \mathcal{N}(0, \sigma^2),
\end{equation}
where
\begin{equation}
\sigma^2 = \left\{ \frac{2}{\pi^2} \int_0^\infty \left[ g(t) \left( \sqrt{1 - \rho^2(t)} + \rho(t) \arctan \frac{\rho(t)}{\sqrt{1 - \rho^2(t)}} - 1 \right) dt + \frac{1}{\pi} \right] \right\} \times (\arctan(2\beta - 1) - \arctan(2\alpha - 1)),
\end{equation}

with
\begin{equation}
g(t) = \frac{1 - (1 + t^2)^{-t^2}}{e} (1 - e^{-t^2})^{3/2} \quad \text{and} \quad \rho(t) = e^{-t^2/2} \frac{1 - t^2 - e^{-t^2}}{1 - e^{-t^2} - t^2 e^{-t^2}}.
\end{equation}

ii. For any given distribution of $a_i$'s,
\begin{equation}
\lim_{n \to \infty} \frac{\mathbb{E}(N_{F_n}\alpha, \beta])}{\sqrt{n}} = \arctan(2\beta - 1) - \arctan(2\alpha - 1).
\end{equation}

iii. If $(a_i) \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$ then provided that $n$ is even or $\arctan(2\beta - 1) - \arctan(2\alpha - 1) \leq \pi/2$, there exists a constant $b_\infty$ as defined in (16) such that
\begin{equation}
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(N_{F_n}\alpha, \beta] = 0) = (\arctan(2\beta - 1) - \arctan(2\alpha - 1)) b_\infty.
\end{equation}

The proof of the main theorem will be presented in Section 2. We exploit the analogous behaviour between the number of roots of Bernstein polynomials and the one of elliptic model and use the existed results for elliptic model.

We fix here some notation. Given a polynomial $f$ and an interval $[a, b] \subset \mathbb{R}$, we denote by $N_f([a, b])$ the number of real roots of $f$ in $[a, b]$. Let $\mathbb{I}_A$ denote the indicator function of a set $A$. If $f$ and $g$ are two real functions, we write $f = O(g)$ if there exists a constant $C > 0$, such that $f(x) \leq C g(x)$ for all $x$; $f = o(g)$ if $f(x)/g(x) \to 0$ as $x \to \infty$. 

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2. Proof of the main theorem

2.1. On number of roots of Bernstein polynomials and the one of elliptic model.
At first sight, one can see the similarity in the variance structure of the coefficients of the random polynomials of two kinds. Moreover, we observe that

\[ F_n(x) = \sum_{i=0}^{n} \sqrt{\binom{n}{i}} a_i x^i (1 - x)^{n-i} = (1 - x)^n f_n \left( \frac{x}{1-x} \right)^i, \]

with \( f_n(x) = \sum_{i=0}^{n} \sqrt{\binom{n}{i}} a_i x^i \) the random elliptic polynomial. Hence for any \( x \neq 1 \),

\[ x \text{ is a root of } F_n \iff \frac{x}{1-x} \text{ is a root of } f_n. \]

Thus for any \( \alpha \leq \beta \),

\[ N_{F_n}[\alpha, \beta] = N_{f_n} \left[ \frac{1-\alpha}{\alpha}, \frac{1-\beta}{\beta} \right]. \tag{7} \]

A useful trick (see [5]) to deal with the elliptic model is to project a homogenization of \( f_n \) on the unit circle \( S^1 \), i.e. consider the following process

\[ G_n(t) = \sum_{i=0}^{n} \sqrt{\binom{n}{i}} a_i (\cos t)^i (\sin t)^{n-i}, \]

with \( t \in [0, \pi] \). It has been shown in [5] that

\[ x \text{ is a root of } f_n \iff \arctan(x) \text{ is a root of } G_n. \]

Thus for any \( a \leq b \),

\[ N_{f_n}[a, b] = N_{G_n}[\arctan(a), \arctan(b)]. \tag{8} \]

It follows from (7) and (8) that

\[ N_{F_n}[\alpha, \beta] = N_{G_n}[\arctan \frac{1-\alpha}{\alpha}, \arctan \frac{1-\beta}{\beta}]. \tag{9} \]

Under the condition \( (a_i) \sim^d \mathcal{N}(0, 1) \), the process \( G_n(t) \) is a centered stationary Gaussian process. Moreover, it is shown in [9] that

\[ \mathbb{E}(N_{G_n}[0, \pi]) = \sqrt{n}. \]

Hence, for any \( [a, b] \subset [0, \pi] \)

\[ \mathbb{E}(N_{G_n}[a, b]) = \sqrt{n} \frac{b-a}{\pi}. \]

Combining this with (9), we get that the expected number of roots of Bernstein polynomials in the interval \( [\alpha, \beta] \) is

\[ \sqrt{n} \left| \arctan \frac{1-\alpha}{\alpha} - \arctan \frac{1-\beta}{\beta} \right| = \frac{\sqrt{n}}{\pi} (\arctan(2\beta - 1) - \arctan(2\alpha - 1)). \]

Then one revisits the result of Armentano and Dedieu [1].
2.2. Proof of CLT. As in [5], we consider the rescaled process:

\[ Z_n(t) = G_n \left( \frac{t}{\sqrt{n}} \right), \]

for \( t \in [0, \pi n] \). It is easy to see that the covariance function of \( Z_n(t) \) is

\[ r_n(t) = \cos^n \left( \frac{t}{\sqrt{n}} \right). \]

Since the process \( G_n(t) \) is a centered stationary Gaussian process, so is the process \( Z_n(t) \). In particular, we have

\[ N_{Z_n}[a, b] \overset{(d)}{=} N_{Z_n}[0, b - a]. \]

It follows from (9), (10) and (12) that

\[ N_{F_n}[\alpha, \beta] \overset{(d)}{=} N_{Z_n}[0, A\sqrt{n}], \]

with

\[ A = \arctan \frac{1 - \alpha}{\alpha} - \arctan \frac{1 - \beta}{\beta} = \arctan(2\beta - 1) - \arctan(2\alpha - 1). \]

In the sequel, we closely follow the arguments of Dalmao [5] using the Stein-Malliavin method. For the simplicity of notation, let us shortly write \( N_{Z_n}[0, A\sqrt{n}] \) by \( N_n \). We first control the variance of \( N_n \).

**Lemma 2.1.** We have

\[ \mathbb{E}(N_n(N_n - 1)) = \frac{2}{\pi^2} \int_0^{A\sqrt{n}} (A\sqrt{n} - t) g_n(t) \left( \sqrt{1 - \rho_n^2(t)} + \rho_n(t) \arctan \frac{\rho_n(t)}{\sqrt{1 - \rho_n^2(t)}} \right) dt, \]

where

- \( g_n(t) = 2\pi p_n(t) v_n(t) \) with
  \[ v_n(t) = 1 - \frac{n \cos^{2n-2} \left( \frac{t}{\sqrt{n}} \right) \sin^2 \left( \frac{t}{\sqrt{n}} \right)}{1 - \cos^{2n} \left( \frac{t}{\sqrt{n}} \right)} \quad \text{and} \quad p_n(t) = \frac{1}{2\pi \sqrt{1 - \cos^{2n} \left( \frac{t}{\sqrt{n}} \right)}}, \]

- \( \rho_n(t) = \cos^{n-2} \left( \frac{t}{\sqrt{n}} \right) \frac{1 - n \sin^2 \left( \frac{t}{\sqrt{n}} \right) - \cos^{2n} \left( \frac{t}{\sqrt{n}} \right)}{1 - \cos^{2n} \left( \frac{t}{\sqrt{n}} \right) - n \cos^{2n-2} \left( \frac{t}{\sqrt{n}} \right) \sin^2 \left( \frac{t}{\sqrt{n}} \right)}. \)

**Proof.** By Kac-Rice formula for second factorial moment of \( N_n \) (see, for instance [3]),

\[ \mathbb{E}(N_n(N_n - 1)) = \int_{[0, A\sqrt{n}]} \mathbb{E}(|Z'_n(t)Z'_n(s)| \mid Z_n(t) = Z_n(s) = 0) p_{Z_n(t), Z_n(s)}(0, 0) dt ds, \]

where \( p_{Z_n(t), Z_n(s)}(0, 0) \) is the value of the joint density function of the Gaussian vector \((Z_n(t), Z_n(s))\) at \((0, 0)\). It is easy to check that:

- \( p_{Z_n(t), Z_n(s)}(0, 0) = p_n(t - s) \).
- \( v_n(t - s) \) is the conditional variance of both \( Z'_n(t) \) and \( Z'_n(s) \) under the condition \( Z_n(t) = Z_n(s) = 0 \).
\( \rho_n(t-s) \) is the conditional correlation between \( Z'_n(t) \) and \( Z'_n(s) \) under the condition \( Z_n(t) = Z_n(s) = 0 \).

From the fact that if \((X,Y) \sim \mathcal{N}(0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix})\) then
\[
E|XY| = \sqrt{1 - \rho^2} + \rho \arctan \frac{\rho}{\sqrt{1 - \rho^2}},
\]
the conditional expectation in the integral is equal to
\[
E (|Z'_n(t)Z'_n(s)| \mid Z_n(t) = Z_n(s) = 0) = v_n(t-s) \left( \sqrt{1 - \rho_n(t-s)^2} + \rho_n(t-s) \arctan \frac{\rho_n(t-s)}{\sqrt{1 - \rho_n^2(t-s)}} \right).
\]

Then the result follows. \( \square \)

**Lemma 2.2.** As \( n \to \infty \), one has
\[
\frac{\text{Var}(N_n)}{\sqrt{n}} \to \frac{2A}{\pi^2} \int_0^\infty \left[ g(t) \left( \sqrt{1 - \rho^2(t)} + \rho(t) \arctan \left( \frac{\rho(t)}{\sqrt{1 - \rho^2(t)}} \right) \right) - 1 \right] dt + \frac{A}{\sqrt{\pi}} < \infty,
\]
where \( g(t) \) and \( \rho(t) \) are defined as in (6).

**Proof.** It is clear that
\[
\text{Var}(N_n) = \mathbb{E}(N_n(N_n - 1)) - (\mathbb{E}N_n)^2 + \mathbb{E}N_n.
\]
Since \( \mathbb{E}N_n = A\sqrt{n}/\pi \),
\[
(\mathbb{E}N_n)^2 = \left( \frac{A\sqrt{n}}{\pi} \right)^2 = \frac{2}{\pi^2} \int_0^{A\sqrt{n}} (A\sqrt{n} - t) dt.
\]
Combining these equations with Lemma 2.1 gives that
\[
\text{Var}(N_n) = \frac{2}{\pi^2} \int_0^{A\sqrt{n}} \left( A\sqrt{n} - t \right) \left[ g_n(t) \sqrt{1 - \rho_n^2(t)} + \rho_n(t) \arctan \left( \frac{\rho_n(t)}{\sqrt{1 - \rho_n^2(t)}} \right) - 1 \right] dt + \frac{A\sqrt{n}}{\pi}.
\]
Therefore
\[
\frac{\text{Var}(N_n)}{\sqrt{n}} = \frac{2A}{\pi^2} \int_0^{A\sqrt{n}} \left( 1 - \frac{t}{A\sqrt{n}} \right) \left[ g_n(t) \sqrt{1 - \rho_n^2(t)} + \rho_n(t) \arctan \left( \frac{\rho_n(t)}{\sqrt{1 - \rho_n^2(t)}} \right) - 1 \right] dt + \frac{A}{\sqrt{\pi}}.
\]
As in [5, Lemma 2], one has the pointwise convergences \( g_n(t) \to g(t) \) and \( \rho_n(t) \to \rho(t) \) and an integrable upper bound for the function inside the integral. Then one can easily deduce the result by using dominated convergence theorem. \( \square \)

Next, as in [2] or [5], one has the Wiener chaos expansion of \( N_n \) as follows.

**Lemma 2.3.** The Wiener chaos expansion of \( N_n \) is
\[
\frac{N_n - \mathbb{E}N_n}{n^{1/4}} = \sum_{q=2}^\infty I_{q,n},
\]
where
\[
I_{q,n} = n^{-1/4} \int_0^{A \sqrt{n}} \left( \sum_{l=0}^{\lfloor q/2 \rfloor} b_{q-2l} a_{2l} H_{q-2l}(Z_n(t)) H_{2l}(Z_n'(t)) \right) dt,
\]
with \( H_k(.) \) the \( k \)-th Hermite polynomial, \( a_{2l} = 2(-1)^{l+1}/(\sqrt{2\pi} 2^l l! (2l-1)) \), and \( b_k = H_k(0)/(\sqrt{2\pi} k!) \).

From the orthogonality of Wiener chaos, one knows that
\[
\frac{\text{Var}(N_n)}{\sqrt{n}} = \sum_{q=2}^{\infty} \text{Var}(I_{q,n}).
\]

Then from Lemma 2.2, the variance of all the chaos are uniformly bounded. Hence to complete the proof of CLT for \( N_n \), by Stein-Malliavin method, one just need to prove the CLT for each chaos. To do so, one fixes a standard Brownian motion \( B \) and expresses \( Z_n(t) = \int_{\mathbb{R}} h_n(t, \lambda) dB(\lambda) \) where
\[
h_n(t, \lambda) = \sum_{i=0}^{n} \sqrt{\binom{n}{i}} \cos(t/\sqrt{n}) \sin^{n-i}(t/\sqrt{n}) \mathbb{E}[i+i+1](\lambda).
\]

Then for the \( q \)-th chaos projection \( I_{q,n} \), the corresponding function \( g_{q,n} \) in the isonormal Hilbert space is
\[
g_{q,n}(\lambda_q) = n^{-1/4} \int_0^{A \sqrt{n}} \left( \sum_{l=0}^{\lfloor q/2 \rfloor} b_{q-2l} a_{2l} \left( h_n^{\otimes q-2l}(t, \lambda_q-2l) \otimes h_n^{\otimes 2l}(t, \lambda_{2l}) \right) \right) dt,
\]
where \( \lambda_i \in \mathbb{R}^1 \); the symbol \( \otimes \) means the tensorial product and \( h_n'(t, \lambda) = \partial_t h(t, \lambda)/\|\partial_t h(t, \lambda)\|_2 \).

As explained in the proof of Proposition 4.1 in [5], the CLT for the chaos projection \( I_{q,n} \) is equivalent to the following lemma.

**Lemma 2.4.** Fix \( q \geq 2 \). For \( g_{q,n} \) as above and each \( k = 1, 2, \ldots, q-1 \), define the \( k \)-contraction
\[
g_{q,n} \otimes_k g_{q,n}(\lambda_{2q-2k}) = \int_{\mathbb{R}^k} g_{q,n}(z_1, \ldots, z_k, \lambda_{q-k}) g_{q,n}(z_1, \ldots, z_k, \lambda_{q-k}') dz_1 \ldots dz_k,
\]
where \( \lambda_{2q-2k} = \lambda_{q-k} \otimes \lambda_{q-k}' \). Then as \( n \to \infty \), this \( k \)-contraction tends to 0.

**Proof.** As Proposition 4.1 in [5], the quadratic norm of the contraction is equal to
\[
\|g_{q,n} \otimes_k g_{q,n}(\cdot)\|^2_2 = \frac{1}{n} \int_{[0,A\sqrt{n}]^4} \sum_{0 \leq j_1, j_2, j_3, j_4 \leq \lfloor q/2 \rfloor} \prod_{l=1}^{4} a_{2j_l} b_{q-2j_l} \frac{1}{q!} \sum_{\sigma \in S_q} \prod_{i=0}^{2} \left( r_n^{(i)}(t - s) \right)^{\alpha_i} \left( r_n^{(i)}(t' - s') \right)^{\beta_i} \left( r_n^{(i)}(s-s') \right)^{\gamma_i} \left( r_n^{(i)}(t-t') \right)^{\delta_i} dt ds dt' ds',
\]
where \( S_q \) is the set of permutations of \( \{1, 2, \ldots, q\} \); and \( \alpha_i, \beta_i, \gamma_i, \delta_i \) depend on \( \sigma \) and \( (j_1, j_2, j_3, j_4) \) satisfying \( \sum_{i=0}^{2} \alpha_i = \sum_{i=0}^{2} \beta_i = k \) and \( \sum_{i=0}^{2} \gamma_i = \sum_{i=0}^{2} \delta_i = q-k \) and some other constraints, and \( r_n^{(i)} \) is the \( i \)-th derivative of \( r_n \).
Note that \( k \geq 1, q - k \geq 1 \) and \( |v_{n,i}(\cdot)| \leq 1 \) for \( i = 0, 1, 2 \) (since \( \text{Var}(Z_n(t)) = \text{Var}(Z'_n(t)) = 1 \)). Therefore one has the following upper bound

\[
\|g_{q,n} \otimes_k g_{q,n}(\cdot)\|^2_2 
\leq \sum_{i_1, \ldots, i_4 \in \{0, 1, 2\}} \frac{(\text{const})}{n} \int_{[0, A\sqrt{n}]^4} [r_{n,i_1}(t - s)r_{n,i_2}(t' - s')r_{n,i_3}(s - s')r_{n,i_4}(t - t')]|dt|\,|ds|\,|dt'|\,|ds'|
\leq \sum_{i_1, \ldots, i_4 \in \{0, 1, 2\}} \frac{(\text{const})}{n} \int_{[0, \pi\sqrt{n}]^4} [r_{n,i_1}(t - s)r_{n,i_2}(t' - s')r_{n,i_3}(s - s')r_{n,i_4}(t - t')]|dt|\,|ds|\,|dt'|\,|ds'|
\leq \sum_{i_1, \ldots, i_4 \in \{0, 1, 2\}} \frac{(\text{const})}{n} \int dt' \int dt \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} |r_{n,i}(x)| \,|r_{n,i}(y)| \,|r_{n,i}(u)| \,|dx| \,|dy| \,|du|.
\]

Again, in [5], the integral \( \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} |r_{n,i}(x)| \,|dx| \) is proven to be uniformly finite. Then the result follows. \( \square \)

2.3. Proof of Universality. By the linearity of expectation, without loss of generality, one just consider the interval \([\alpha, \beta] \subset (0, 1)\). We recall a universal result for the expectation of the number of roots of elliptic model due to Tao and Vu.

**Lemma 2.5.** [21, Theorem 5.6] Let \( f_n(x) = \sum_{i=0}^{n} \sqrt{\binom{n}{i}} a_i x^i \) be the random elliptic polynomial, with \( \{a_0, a_1, \ldots\} \) a sequence of i.i.d random variables of zero mean, unit variance, finite \((2 + \epsilon)\)-moment for some \( \epsilon > 0 \), satisfying \( \mathbb{P}(a_i = 0) = 0 \). Then there exists a positive constant \( c \) depending on \( \epsilon \) and the distribution of the coefficients such that for any interval \([a, b]\),

\[
\mathbb{E}(N_{f_n}[a, b]) = \int_a^b \frac{dx}{\pi(1 + x^2/n)} + O(n^{1/2-\epsilon}).
\]

Using the above lemma we have

\[
\mathbb{E}\left(N_{f_n}\left[\frac{1 - \alpha}{\alpha}, \frac{1 - \beta}{\beta}\right]\right) = \frac{(1-\alpha)/\alpha}{\pi(1 + x^2/n)} + o(n^{1/2})
\]

\[
= \frac{\arctan(2\beta - 1) - \arctan(2\alpha - 1)}{\pi} \sqrt{n} + o(n^{1/2}).
\]

Combining this equation with (7), we get Part (ii) of Theorem 1.1.

2.4. Proof of Persistence probability. From (13), it is clear that

\[
\mathbb{P}(N_{f_n}[\alpha, \beta] = 0) = \mathbb{P}(Z_n(t) \text{ has no root in the interval } [0, A\sqrt{n}])
\]

\[
= 2\mathbb{P}(Z_n(t) > 0, \forall t \in [0, A\sqrt{n}]).
\]

Recall that for \( n \) even or \( \arctan(2\beta - 1) - \arctan(2\alpha - 1) \leq \pi/2 \), the covariance function of the Gaussian process \( Z_n(t) \), that is \( r_n(t) = \cos^n \left( \frac{t}{\sqrt{n}} \right) \), is nonnegative and converges.
to the corresponding one \( r_\infty(t) = e^{-t^2/2} \) of a centered stationary Gaussian process \( Z_\infty(t) \).

Then from Theorem 1.6 and Remark 1.9 in [6], one can deduce that

\[
\lim_{n \to \infty} \frac{1}{A\sqrt{n}} \log P \left( Z_n(t) > 0, \forall t \in [0, A\sqrt{n}] \right) = b_\infty,
\]

where \( b_\infty \) is defined as the limit

\[
b_\infty = \lim_{T \to \infty} \frac{1}{T} \log P \left( Z_\infty(t) > 0, \forall t \in [0, T] \right),
\]

which exists and is finite. The result follows from (14), (15) and (16).

REFERENCES
